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ON THE AXIOMATIC FOUNDATIONS OF DIMENSIONAL ANALYSIS

by

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This paper reformulates the basic axioms of dimensional analysis so as to clear up some deeply entrenched misconceptions relating to the nature of physical dimensions and generalized units. Certain novel relations are thereby discovered which although unorthodox are nevertheless useful and correct. It is shown how various systems of natural units can be constructed so as to embody the same logical structure that characterizes some given system of fixed units. The important fact is brought out that every physical

equation which is valid in some given system of units remains valid if all quantities be converted into any other system which embodies the same logical structure. The paper also shows that the dimensionless pi's of Buckingham's Pi Theorem simply represent various physical parameters as expressed in some appropriate system of consistent natural units. The fundamental dimensional principles considered in this paper apply in some form to every quantitative analytical and experimental problem in the entire realm of physical science and engineering.

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ON THE AXIOMATIC FOUNDATIONS OF DIMENSIONAL ANALYSIS

1. Introduction

While the broad principles of dimensional analysis have long been familiar to scientists and engineers, there nevertheless still persists a significant degree of confusion and inconsistency relating to the axiomatic foundations of the subject. This confusion constitutes an obstacle to the fuller utilization of the power of dimensional methods in scientific analysis. The aim of this paper is to clarify the key issues involved and to resolve them. This requires that a careful distinction be made between the concept of a physical dimension on the one hand and the associated concept of a generalized unit on the other, and that the contingent nature of the relationship between these two kinds of entities be made explicit. This paper also clarifies the important and familiar concept of a dimensionless quantity including the fact that the apparent non-dimensionality of such a quantity is actually dependent on the logical structure of the system of units in which it happens to be expressed; for example, specific heat turns out to be dimensionless if expressed in consistent units of a particular type, but not if expressed in those of another type.

The reader is assumed to have a general working knowledge of units and dimensions, consequently routine aspects of the subject are passed over without redundant explanations.

2. Consistent Units

The equations of physics are customarily written in terms of consistent units. This means that a number of types of physical quantities are selected

as fundamental and the sizes of the corresponding units are prescribed arbitrarily; then the size of every other kind of unit is so determined as to make it consistent with the prescribed magnitudes of the fundamental units, according to certain logical rules.

There is some latitude in the choice of the fundamental quantities. For the sake of definiteness, the fundamental units are here taken initially as those of force, mass, length, time, heat, temperature and electrical charge. We denote these fundamental units by the mnemonic symbols F, M, L, T, H, θ, and Q, respectively.

These seven fundamental units are ample to cover a tremendous range of physical phenomena. In most fields of application, not all of them are required. For example, the field of geometry demands only one fundamental dimension, namely L. Statics requires two, L and F. Kinematics also requires two, L and T. Dynamics involves F, M, L and T. To these four, the field of theromodynamics adds the units H and θ.

Under certain conditions, the number of fundamental dimensions required in a given field may be further reduced. Also the identities of the particular units which are designated as fundamental may be changed. These matters are taken up later in this paper.

We term the symbols F, M. L, etc., generalized units because their respective magnitudes are not necessarily fixed once and for all, but may be assigned in some appropriate way according to the context in a given problem or field of application. In fact generalized units are classified in this paper into two types, fixed and natural. Fixed units are those whose magnitudes have been established permanently by international agreement, and which are very precisely defined in terms of appropriate invariant

physical standards. Thus the kilogram and the meter are typical fixed units. Natural units, on the other hand, are those whose magnitudes are defined in some convenient and significant way in relation to some physical system or phenomenon that happens to be of interest. For example, in connection with the performance analysis of a certain aircraft, the total mass and wingspan of the craft might be defined as the appropriate natural units of mass and length, respectively.

In the discussion which follows we are often concerned with some specified set of n units selected from some standard system of consistent fixed units. It is convenient to denote the units in such a set by the symbols U_1, U_2, U_3, ---U_i ---U_n. Sometimes we are interested also in a corresponding set of natural units which are related to the above set of fixed units in a particular way which is explained later. In such cases we denote these natural units by the corresponding starred symbols U_1^*, U_2^*, U_3^* ---U_i^* ---U_n^*. Moreover, it is usually convenient to define the actual magnitudes of the natural units, which may vary from one problem or application to another, as numerical multiples of the known magnitudes of the corresponding units in the fixed system. For this purpose we introduce the following nomenclature, namely,

$$\begin{aligned} U_i^* &= N_i\, U_i \\ i &= 1,2,3,\text{---}n \end{aligned} \qquad (2.1)$$

In this connection it should be recognized that the symbols N_i (i = 1,2, 3,---n) now represent ordinary numbers. This is in contrast with the symbols U_i and U_i^* which themselves can never be meaningfully reduced to or represented by pure numbers. The fact that numbers can be substituted for the coefficients N_i in Eqs (2.1) is useful later in simplifying the form and the interpretation of certain analytical results.

It has already been remarked that the magnitude of each derived unit must be defined in such a way as to remain consistent with the fundamental units, irrespective of how the magnitudes of the latter may ultimately be prescribed. This relation of consistency for each derived unit can be expressed in any one of three alternative but equivalent forms. The first is simply in the form of a word definition, the second is in the form of an ordinary mathematical equation, the third is in the form of a symbolic relation among the various generalized units which are involved.

By way of example, consider the consistent units of area and of velocity, respectively. The relations of consistency may be summarized as follows.

Example 1: Area

a. Word definition -

"The consistent derived unit of area is the area of a square of unit length on a side."

b. Mathematical equation -

$$A = b^2 \tag{2.2}$$

where A = area of square

b = length of side

c. Symbolic relation among generalized units -

$$U(A) = L^2 \tag{2.3}$$

where $U(A)$ = unit of area

L = unit of length

Example 2: Velocity

a. Word definition -

"The consistent derived unit of velocity is the velocity that corresponds to unit length traversed per unit time".

b. Mathematical equation -

$$v = \left(\frac{dx}{dt}\right)$$

where v = velocity

x = displacement

t = time

c. Symbolic relation among generalized units -

$$U(v) = LT^{-1} \tag{2.5}$$

where $U(v)$ = unit of velocity

L = unit of length

T = unit of time

The number of such examples could be extended indefinitely. Notice that if any one of these three basic forms of a statement of consistency be valid, the other two forms must necessarily be valid as well. For the sake of conciseness, no further rules of consistency in verbal form are given at this point, but the reader should realize that such a statement is always possible in principle and that it is sometimes very helpful for clarity. The third method of expressing relations of consistency, namely, the method of using exponential relations among the various generalized units involved, is the mode of greatest interest and value in dimensional analysis and it is the one primarily used in this paper.

Many writers assume that the meaning of these exponential relations is self-evident. As a matter of fact, use of such symbolic relations always implies a certain postulate which may or may not be self-evident but which in all events is well worth pointing out explicitly.

For this purpose we first note that the magnitude of any dimensional quantity can be specified quantitatively only in the form of a numerical

multiple of some appropriate unit. This numerical multiple, or the algebraic symbol which denotes it, is termed the measure. Thus unit and measure always constitute an inseparable pair.

The equations of physics are ordinarily written in terms of symbols which represent the numerical measures of the corresponding quantities in some stated or understood system of consistent units. We may term these the equations of measure.

Now the postulate in question simply asserts that if we complete each equation of measure by explicitly including with each quantity of measure its corresponding generalized unit, the completed equation so obtained continues to be valid and to satisfy all of the ordinary rules of mathematics! This postulate converts all of the generalized units from mere identifying labels to the status of genuine mathematical symbols.

To illustrate this idea, let us apply it to the mathematical expression used above in connection with the definiton of the consistent unit of velocity. Thus for the expanded expression we obtain

$$vU(v) = \frac{d(xL)}{d(tT)} \tag{2.6}$$

while for the original equation of measure we have as before

$$v = \frac{dx}{dt} \tag{2.7}$$

Dividing Eq (2.6) by (2.7) gives

$$U(v) = LT^{-1} \tag{2.8}$$

which shows that associated with the relation of measure Eq (2.7) there exists a corresponding relation of units, Eq (2.8). As a result of the foregoing postulate, every equation of measure in physics has its accompanying relation

of units or consistency relation. These relations may be regarded as symbolic statements of equivalence. They assert that the symbol which represents a given consistent derived unit can be replaced by some specified exponential combination of the symbols which represent various fundamental units and that such a substitution never leads to any mathematical inconsistency in the representation. Notice, however, that this symbolic substitution is valid if and only if the actual magnitude of the derived unit in question is properly related to the assigned magnitudes of the fundamental units as explained earlier. Thus, the exponential relations of consistency, while symbolic in nature and themselves incapable of being reduced to pure numbers, are nevertheless strictly quantitative in their essential meaning and implications.

A representative sample of consistent derived units is listed in Table 2.1. The appropriate symbolic relations of consistency among the generalized units are summarized in the column marked "Basic System". The reader may if he wishes supply corresponding word definitions and definitions in terms of ordinary mathematical expressions of measure. The meaning of the other columns in Table 2.1 is discussed later.

A curious feature of the foregoing symbolic consistency relations is that the expressions themselves are (for the most part) old and familiar, yet the interpretations that are placed upon them are varied and conflicting. In some cases no real interpretation is offered on the grounds, presumably, that the meaning should be self-evident. This assumption would be more convincing if all writers who do offer an interpretation were to provide the same one, or at least equivalent ones, but in fact they do not.

To shed further light on these ambiguities, we now introduce an important distinction in terminology beyween two related concepts, namely, the concept of a generalized unit and the concept of a dimension. The generalized unit has already been amply explained; the significant point to note in the present context is that the generalized unit, besides having its own characteristic qualitative character, is also quantitative and mathematical in nature.

In contrast, the concept of a physical dimension is here defined to be purely qualitative and devoid of all quantitative significance. Thus, the distinct qualitative categories denoted by words like force, length, time, viscosity, momentum, magnetic flux, temperature gradient, and so on can be said to represent corresponding physical dimensions. Whenever any quantitative or mathematical relation is involved, however, each physical dimension must be represented by its appropriate generalized unit. In other words the term dimension identifies an entity primarily as being of a certain physical kind whereas the term generalized unit identifies it primarily as having a certain definite magnitude.

For the most part, each physical dimension is represented by its characteristic generalized unit, like force by F, length by L, time by T, and so forth. If matters were always so simple, the distinction we are here making would be redundant and unnecessary. It happens, however, that two (or more) distinct physical quantities are sometimes represented by one and the same generalized unit! For example, both work and torque which are physically quite different, are measured in terms of the generalized unit FL. Similarly, specific heat and specific entropy which again are physically different, are both measured by the generalized unit $H/M\Theta$.

Moreover, in certain kinds of systems the pound unit is used in reference both to the physical dimension of mass and to the altogether dissimilar physical dimension of force. To make matters worse, the appropriate generalized unit for measuring any specified physical dimension depends entirely on the logical structure of the selected system of units. By logical structure we refer to the particular set of consistency rules which characterize the system; these are subject to a certain amount of variability. Moreover, the apparent symbolic form of the generalized unit which corresponds to a given physical dimension depends, of course, on the particular units which are chosen as fundamental; here again one encounters a certain amount of variability.

The failure to distinguish clearly between physical dimensions and generalized units leads to considerable confusion in cases like these. Many writers use these terms and the corresponding concepts as if they were interchangeable. The symbolic consistency relations like those of Table 2.1 are often interpreted as if the symbols F, M, L, etc. represent qualitive dimensions rather than quantitative units. This practice so beclouds the significance of the relations that it becomes difficult to say whether any real significance survives. Moreover, such an unconscious confounding of dimension with generalized units is misleading in yet another way: inasmuch as the nature of each dimension is an unchanging physical reality the mis-identification of the dimension with the generalized unit falsely suggests that the relations that subsist among the latter are also universal and unchanging when in fact these relations are strictly contingent on the logical structure of the particular system of units under consideration. Thus it is not uncommon to read or hear the statement that "force has the dimensions of mass times acceleration" stated as if this

were some immutable natural law. What this statement really should say is that, owing to the importance of Newton's second law of motion, it is often convenient to adopt as a consistent derived unit of force that force which imparts unit acceleration to unit mass. What the original statement totally overlooks is the fact that it is always possible and sometimes convenient to drop this particular consistency rule and to replace it with another, or perhaps with none at all!

The discussion has shown that the magnitudes of all units in a given system is fixed by three factors. These are firstly, the identity of the selected fundamental units, secondly, the magnitudes that happen to be assigned to them, and thirdly, the rules of consistency which relate the magnitudes of the derived units to those of the fundamental ones.

In this connection it is of the greatest value and importance to identify the invariants associated with such a system. By an invariant is meant any significant quantity, aspect or feature of the situation which is independent of how the actual magnitudes of the fundamental units happen to be assigned.

A little reflection shows that there are at least two invariant features of great significance. One of these is the simple fact that the mathematical form of every valid physical equation is totally independent of the magnitudes that happen to be assigned to the fundamental units! This important result requires no separate proof since it follows at once from the very nature of the consistency relations themselves.

Of course, if one changes the sizes of the fundamental units, then the sizes of the consistent derived units change also, and the numerical values of all quantities of measure in every equation change accordingly. However, the analytical form of every valid equation remains totally unaffected and the new numbers still satisfy the original equations. In particular, no new

conversion factors of any sort need be inserted into the equations in connection with any such shift despite the fact that the magnitudes of all units may be changed drastically!

It is extraordinary that a fact of such fundamental significance as the foregoing should continue to escape the general attention or interest of the scientific community. One of the basic purposes of the present paper is simply to call attention to this important rule, namely, that all physical equations are invariant with respect to changes in the magnitudes of the fundamental units!

A second important invariant in this situation is represented by the familiar concept of a dimensionless number. In most cases we recognize a dimensionless number in practice by the fact that there are no units associated with it. In more complicated cases units may at first seem to be involved, but they are ultimately found to cancel out identically. If dimensional rules and operations were strictly consistent, such cancellation of units would always occur for every dimensionless quantity and there would never be any difficulty in determining whether a given quantity is really dimensionless or not. In practice, however, unit symbols are employed partly in the quantitative role of generalized units and partly in the purely qualitative role of dimensional labels; moreover, the two senses are seldom clearly distinguished. As a consequence, it can and does happen that a quantity may be labeled with apparent units yet turn out to be dimensionless in fact. We shall refer to any units used in this way as being quantitatively redundant. Nevertheless it should not be overlooked that the persistence of such usages arises from the fact that the pseudo-units involved,

altho without any quantitative significance whatever, may convey purely qualitative information that is deemed useful. Of course, in the context of the present discussion which aims at uncovering and displaying the axiomatic foundations of dimensional analysis in their simplest and starkest quantitative terms, all redundant symbols must be resolutely discarded.

In order to get around the above difficulties in the identification of certain dimensionless numbers, it is necessary to adopt a somewhat more elaborate definition as follows: a dimensionless quantity is any quantity whose numerical measure is independent of the actual magnitudes assigned to the fundamental units (in a dimensional system having a prescribed set of consistency relations).

Notice that this definition makes the issue of non-dimensionality depend in part on the particular consistency rules which characterize the system. Thus a quantity having some prescribed physical dimension may have definite physical units if prescribed in one type of system but be truly dimensionless if prescribed in another. Good examples of this are specific heat and specific entropy which may be either dimensional or dimensionless depending on the type of system involved. This example is analyzed in more detail later.

Of course the great importance of dimensionless quantities in science stems precisely from their characteristic property of invariance. It is therefore all the more unfortunate that the contingent nature of this important attribute is not more widely recognized.

3. Inertial Units of Force and Mass

Consider Newton's second law of motion as it would be written in any system of units in which all four of the generalized units F, M, L and T

could be specified arbitrarily. Let f denote the force acting on a body of mass m, and let a denote the corresponding acceleration of the body. Newton's law would then assume the form

$$f = k_I \, m \, a \tag{3.1}$$

where k_I (the inertial constant) is a universal constant, that is, a constant whose numerical measure depends only on the relative magnitudes of the relevant generalized units and on nothing else. In this case the relevant generalized units are, of course, F, M, L and T.

The corresponding relation of consistency among the various units involved would then take the form

$$F = U(k_I) \; M \; L \; T^{-2} \tag{3.2}$$

where symbol $U(k_I)$ denotes the generalized units of the dimensional constant k_I.

Owing to the fact the Newton's second law is applicable over such a tremendous range of scientific and engineering problems, it becomes attractive to incorporate it as a consistency rule into the structure of the system of units. We term any system of units constructed in this way an inertial system.

Stated in words, the consistency rule takes the following form: the consistent inertial unit of force is the force that imparts unit acceleration to unit mass.

An equivalent statement is the following: the consistent inertial unit of mass is the mass that sustains unit acceleration under the action of unit force.

In either case, and regardless of how the magnitudes of the three

fundamental units be specified (that is, the magnitudes of M, L, T or else the magnitudes of F, L, T as the case may be), it follows that

$$k_I = 1 \tag{3.3}$$

and

$$U(k_I) = 1 \tag{3.4}$$

This last equation simply asserts that in any inertial system of units the constant k_I is a dimensionless quantity. This follows at once from the earlier definition of a dimensionless quantity and from the fact that k_I now retains a numerical value of unity irrespective of how the magnitudes of the three fundamental quantities happen to be prescribed.

Consequently Newton's second law now becomes simply

$$f = m\,a \tag{3.5}$$

and the corresponding relation of units becomes

$$F = M\,L\,T^{-2} \tag{3.6}$$

or alternatively

$$M = F\,L^{-1}\,T^{2} \tag{3.7}$$

Eqs (3.6) or (3.7) are among the most commonly encountered relations in all of dimensional analysis. Unfortunately, these relations are often stated as if they represented universal laws. Hence it is necessary to emphasize not only that these expressions are valid for every inertial system whatever the specific units, but also that they are totally meaningless in relation to any system which happens not to be inertial!

It is now apparent that for any inertial system of units, Eq (3.6) expresses F as the consistent derived unit while Eq (3.7) expresses M as the consistent derived unit. Consequently, either F or M, as preferred, may be eliminated from the list of fundamental units.

It happens that in the metric MKS (meter, kilogram, second) system, the customary choice of fundamental units is M = 1 kg, L = 1 m, T = 1 sec with the consistent derived unit being F = 1 newton. On the other hand in the English FPS (foot, pound, second) system, the usual choice of fundamental units is F = 1 lb, L = 1 ft, T = 1 sec with the consistent derived unit being M = 1 slug.

4. Gravitational Units of Force and Mass

Again assuming that all four of the generalized units F, M, L and T may be chosen arbitrarily, let us apply Newton's second law to a body in a gravitational field. In this case acceleration a becomes equal to the gravitational acceleration g, and the applied force f becomes equal to the body weight w. Moreover, inasmuch as the earth's gravity g varies from point to point on the earth's surface, let us stipulate some carefully standardized value g_o and let the coresponding weight be termed the standard weight w_o. Then with only a slight rearrangement in the order of the factors, Newton's law reduces to the form

$$w_o = (k_I \, g_o) \, m \tag{4.1}$$

Now we adopt the following definition of a gravitational system: the consistent gravitational unit of force is the weight of unit mass in the earth's gravitational field (under prescribed standard conditions of gravitational acceleration).

It is at once evident from this definition that

$$k_I \, g_o = 1 \tag{4.2}$$

regardless of the magnitude assigned to M and therefore that

$$U(k_I \, g_o) = 1 = \text{dimensionless!} \tag{4.3}$$

Consequently Eq (4.1) reduces simply to

$$w_o = m \tag{4.4}$$

and the corresponding consistency relation among generalized units becomes

$$F = M \tag{4.5}$$

This last result may come as something of a shock to the individual accustomed to thinking of symbols F and M as representing dimensions rather than generalized units because from that viewpoint Eq (4.5) would seem to imply that "the dimension of force is equal to the dimension of mass", a proposition that defies rational interpretation. Actually Eq (4.5) implies no such thing. It simply asserts that in any gravitational system, the choice of the magnitude of unit M likewise fixes the corresponding magnitude of unit F (or vice versa).

Eq (4.5) simply states in generalized terms what we know to be true in any specific case of a gravitational system, namely, that the force and mass units have a certain duality such that a single unit label pertains to both physical dimensions. Thus we have pound force and pound mass, kilogram force and kilogram mass, and so forth. What the earlier analysis makes clear is that it is only the unit labels like pound or kilogram that have any quantitative significance. If the qualifying terms "force" or "mass" are appended to the unit labels it is only to provide auxiliary qualitative descriptions. Such appendages, if used, are quantitatively redundant.

These observations suggest that it would be appropriate to coin some new symbol, say G for "gravitational unit", which is entirely neutral in its connotation with respect to force or mass, and which could therefore be used to represent either dimension as appropriate in any given case. Thus symbol G becomes the generalization of terms like pound or kilogram which may represent either force or mass according to circumstances. Hence we may rewrite Eq (4.5) in the modified form

$$F = G = M \tag{4.6}$$

Naturally the use of the gravitational unit G in such a dual capacity is appropriate only in a gravitational system of units.

It now follows from (4.2) that for any gravitational system of units, the inertial constant k_I in Newton's second law takes on the numerical value

$$k_I = \frac{1}{g_o} \tag{4.7}$$

Then from Eq (4.3) we may infer that the generalized units of $k_I^{-1} = g_o$ are simply

$$U(g_o) = L\,T^{-2} \tag{4.8}$$

By utilizing Eq (4.7) we obtain Newton's second law for any gravitational system of units in the familiar form

$$f = \frac{1}{g_o}\ m\ a \tag{4.9}$$

The orthodox method of stating the units of g_o in Eq (4.9) is slightly different from that given above. If we distinguish between units F and M, Eq (4.9) implies that

$$U(g_o) = (M\,F^{-1})\ (L\,T^{-2}) \tag{4.10}$$

which is the orthodox way of designating these units. We note, however, that for any gravitational system of units

$$M\,F^{-1} = G\,G^{-1} = 1 = \text{dimensionless!} \tag{4.11}$$

so that the nominal factor $M\,F^{-1}$ is quantitatively redundant.

The standard acceleration of gravity has the value

$$g_o = 9.80665\ \text{m/sec}^2$$

$$= 32.1739\ \text{ft/sec}^2 \tag{4.12}$$

Of course the numerical measure of g_o depends on the magnitudes of units L and T but remains independent of the redundant quantity $F M^{-1}$. Conversely, the quantitative equivalence between F and M as expressed in Eq (4.5) is totally unrelated to the magnitudes which happen to be assigned to the units L and T.

5. Mechanical Units of Energy

The consistent mechanical unit of energy, that is the consistent unit of work U(w), is defined simply as the work done by unit force over unit displacement. This may be written

$$U(w) = F\ L \tag{5.1}$$

The forms of energy we call work and heat are related to each other by the first law of thermodynamics. This may be expressed in the form

$$\oint dq = \frac{1}{J} \oint dw \tag{5.2}$$

where $\oint dq$ denotes the net heat received by a closed system over a thermodynamic cycle, $\oint dw$ denotes the net work done by the system and the factor J, Joule's constant, is a universal constant whose numerical value depends only on the units in which heat and work are expressed. Let H denote the unit of heat.

The relation of units corresponding to Eq (5.2) is

$$H = \frac{FL}{U(J)} \tag{5.3}$$

where U(J) denotes the units of J.

Owing to the tremendous range of applicability of the first law, it is attractive to employ it as a relation of consistency of the dimensional system. We can do so by choosing H as a consistent derived unit such that

$$J = 1 \tag{5.4}$$

and

$$U(J) = 1 = \text{dimensionless} \tag{5.5}$$

whereupon Eqs (5.2) and (5.3) reduce simply to

$$\oint dq = \oint dw \tag{5.6}$$

and

$$H = FL \tag{5.7}$$

In this paper we describe as mechanical any system of units in which heat and all other forms of energy are consistently expressed in this way, that is, exclusively in terms of the work unit FL.

6. Calorimetric Units of Energy

Historically, the conventional units of heat (the calorie, the kilocalorie, the British thermal unit) are derived not from the first law of thermodynamics, but rather from the practice of calorimetry.

The calorimetric equation for water may be written in the form

$$dq = c_{pw}\ m\ dT \tag{6.1}$$

where dq is some small quantity of heat received by mass m of water, dT is the corresponding small temperature rise and c_{pw} denotes the specific heat of water (at some prescribed standard pressure and temperature).

The corresponding relation of units becomes

$$H = U(c_{pw})\ M\ \theta \tag{6.2}$$

The consistent calorimetric unit of heat may now be defined as the quantity of heat that produces unit change of temperature in unit mass of water (at prescribed standard conditions of pressure and temperature).

This definition establishes that at the prescribed standard conditions

$$C_{pw} = 1 \tag{6.3}$$

by definition, irrespective of the magnitudes assigned to the fundamental units M or θ. This means in turn that

$$U(c_{pw}) = 1 = \text{dimensionless} \tag{6.4}$$

Consequently, from Eqs (6.2) and (6.4) the consistency relation for calorimetric units of energy reduces to

$$H = M\,\theta \tag{6.5}$$

This result, while rigorously correct and completely consistent with the axioms used throughout this paper, is quite unorthodox in dimensional analysis. To bring out this aspect, we apply Eq (6.5) to metric and English units as follows.

$$\begin{aligned} 1\ \text{Kcal} &= 1\ \text{Kg}\ {}^{\circ}\text{C} \\ 1\ \text{Btu} &= 1\ \text{lb}\ {}^{\circ}\text{F} \end{aligned} \tag{6.6}$$

This result also explains why in any calorimetric system of units, the quantities specific heat and specific entropy, with apparent units of $H/M\theta$ are actually dimensionless; these units are of course redundant in this case. On the other hand in a mechanical system of units, specific heat and specific enthalphy become true dimensional quantities and the units $H/M\theta$ once more acquire quantitative significance.

Somewhat similar considerations apply to Joule's constant. From Eqs (5.3) and (6.5) we see that in any calorimetric system of units, the units of Joule's constant become

$$U(J) = FL/M\theta \tag{6.7}$$

Further reduction is possible in these cases, the exact form of which depends on whether the system, besides being calorimetric, is either inertial or gravitational. For example, the numerical values of Joule's constant as

expressed in various conventional and unconventional metric and English units are

$$J = 4186 \text{ joule/k cal} = 4186 \text{ newt. m./kg } {}^{\circ}C \tag{6.8}$$

and

$$J = 778.3 \text{ ft lb/Btu} = 778.3 \text{ ft lb/lb } {}^{\circ}F \tag{6.9}$$
$$= 778.3 \text{ ft/}{}^{\circ}F$$

7. Reduced Relations of Consistency

Typical relations of consistency which characterize certain principal types of dimensional systems are illustrated in Table 2.1. The relatively unrestricted basic system in which all seven of the units F, M, L, T, H, θ and Q may be specified arbitrarily has been discussed earlier. Also shown in Table 2.1 are the two important cases of the inertial-mechanical type of system and of the gravitational-calorimetric type. Electrical charge Q has been deleted from both of these latter cases to simplify the subsequent discussion which is limited to non-electrical applications.

The relations of consistency which characterize the inertial mechanical type of system as illustrated in Table 2.1 are very widely used in dimensional analysis. In fact, some texts give the misleading impression that these are the only possible relations. The usefulness of the inertial-mechanical system stems from the fact that it incorporates within its logical structure both Newton's second law of motion and the first law of thermodynamics. In the absence of electrical phenomena, electrical charge Q may be deleted and the number of fundamental dimensions then reduces to four. If English units are used, it is customary to take F, L, T and θ as fundamental. If metric units are employed, units M, L, T and θ are generally chosen as fundamental. For the sake of simplicity, only the F, L, T, θ choice is displayed in Table 2.1.

The convenience of the gravitational-calorimetric system of units stems from the fact that a tremendous body of numerical data, especially in thermodynamics, happens to be expressed in this system. Again excluding electrical charge Q, we once more recover just four fundamental units, namely, G, L, T and θ.

Once the axiomatic principles expounded in this paper have been clearly grasped, it becomes an elementary matter to apply them to any other type of dimensional system which may happen to be useful in a particular context, whether orthodox or unorthodox.

Various unorthodox systems may prove advantageous in connection with special classes of problems. While the detailed discussion of such cases lies outside the scope of this paper, it is instructive to illustrate the concept here briefly with a single arbitrary example. Suppose we define a consistent derived unit of length as that length which causes the standard acceleration of gravity to assume unit measure irrespective of how the magnitude of the unit of time be specified. Then

$$g_o = 1 \tag{7.1}$$

and

$$U(g_o) = L\,T^{-2} = 1 = \text{dimensionless!} \tag{7.2}$$

whereupon

$$L = T^2 \tag{7.3}$$

This example serves to demolish the stereotyped and false concept that unit L must necessarily be independent of unit T; such independence happens to be the case in conventional systems but only by deliberate design. Incidentally, this particular unorthodox dimensional system has the further

interesting property that if the consistent unit of force then be defined on a gravitational basis, it also turns out to be consistent on an inertial basis as well!

8. Change of Base. Natural Units

For the sake of definitness, the discussion beyond this point is restricted specifically to the basic inertial - mechanical type of system with F, L, T and θ initially designated as the four fundamental units. All units are specified initially in some standard fixed system of known units such as the FPS, MKS or CGS systems. In this section we consider the problem of constructing some related natural system of units on the basis of any suitable set of four arbitrary but known physical parameters which are selected as representative of some physical system or phenomenon of interest.

A slight change in notation proves convenient. Let the four fundamental units be rewritten according to the following scheme.

$$\begin{aligned} F &= F_1 = U_1 \\ L &= F_2 = U_2 \\ T &= F_3 = U_3 \\ \theta &= F_4 = U_4 \end{aligned} \tag{8.1}$$

Let all of the remaining (n-4) derived units which happen to be of interest be denoted by the remaining sequence U_5, U_6, --- U_i, --- U_n.

Now the complete set of n consistency relations which characterize the given system of units can be summarized in the form

$$U_i = F_1^{a_{i1}} \; F_2^{a_{i2}} \; F_3^{a_{i3}} \; F_4^{a_{i4}} \qquad i = 1, 2, \text{----} \; n \tag{8.2}$$

where the exponents a_{ij} constitute an n x 4 array of known numerical constants. This array, which we term the consistency matrix, fully defines the essential logical character of the dimensional system under consideration. An example of such an array is given in Table 8.1.

Now consider some related system of natural units which has exactly the same consistency matrix as the above fixed system, but which may differ from the fixed system in the respective magnitudes of its four fundamental units, and therefore in the respective magnitudes of all its units.

For the natural system, the relations which correspond to Eqs (8.2) become

$$U_i^* = F_1^{*\,a_{i1}} \; F_2^{*\,a_{i2}} \; F_3^{*\,a_{i3}} \; F_4^{*\,a_{i4}} \qquad i = 1, 2, \text{---}\ n \tag{8.3}$$

However, the magnitude of each natural unit can also be expressed as some numerical multiple of the magnitude of the corresponding fixed unit. Thus let

$$\begin{aligned} U_i^* &= N_i \, U_i \qquad & i &= 1, 2, \text{---}\ n \\ F_j^* &= M_j \, F_j \qquad & j &= 1, 2, 3, 4 \end{aligned} \tag{8.4}$$

By substituting Eqs (8.4) into (8.3) then dividing thru by (8.2) we obtain the useful result that

$$N_i = M_1^{a_{i1}} \; M_2^{a_{i2}} \; M_3^{a_{i3}} \; M_4^{a_{i4}} \qquad i = 1, 2, \text{---}\ n \tag{8.5}$$

The great practical advantage of Eq (8.5) over the previous expressions is that all of the symbols denoting generalized units have been eliminated from (8.5) and have been replaced by symbols that represent simple numbers!

This not only lowers the level of abstraction involved but also enables us ultimately to solve the resulting equations by normal numerical methods.

It is useful to rewrite Eqs (8.5) in logarithmic form as follows

$$\log N_i = \sum_{j=1}^{4} a_{ij} \log M_j \tag{8.6}$$

Next we introduce matrix notation. Let the logarithmic terms be represented by the following vectors.

$$\begin{Bmatrix} \log M_1 \\ \log M_2 \\ \log M_3 \\ \log M_4 \end{Bmatrix} = \begin{Bmatrix} \log M \end{Bmatrix} \tag{8.7}$$

$$\begin{Bmatrix} \log N_1 \\ \log N_2 \\ ----- \\ \log N_i \\ ----- \\ \log N_n \end{Bmatrix} = \begin{Bmatrix} \log N \end{Bmatrix} \tag{8.8}$$

Let the consistency matrix be represented as follows.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ -- & -- & -- & -- \\ a_{i1} & a_{i2} & a_{i3} & a_{i4} \\ -- & -- & -- & -- \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} \end{bmatrix} = [a] \tag{8.9}$$

With this notation, Eq (8.6) can be rewritten in the concise form

$$\left\{\log N\right\} = [\,a\,]\left\{\log M\right\} \tag{8.10}$$

This important result can now be interpreted as follows. Recall that the magnitudes of the fixed units are known and that the consistency matrix [a] is also known. Now let the magnitudes of the four fundamental units F_1^*, F_2^*, F_3^*, F_4^* in the natural system be specified arbitrarily. In that case the four numbers M_1, M_2, M_3, M_4 are fixed thereby, according to Eqs (8.4). Consequently Eq (8.10) then fixes all n elements of vector $\{\log N\}$. But again by Eq (8.4) this fixes the magnitudes of all the units in the natural system.

Moreover, although corresponding units in the fixed and natural systems may differ drastically, both systems nevertheless embody an identical logical structure. Any physical equation valid in either system is also valid in the other.

Eq (8.10) also brings out the fact that for any known system of fixed units having some specified consistency matrix [a], there exist an infinite number of possible logically similar natural systems corresponding to the fourfold infinity of possible choices for the vector $\{\log M\}$. It is sometimes convenient to refer to this entire hypothetical set, that is, to the set of all possible systems of natural units which are logically equivalent to some given fixed system, as constituting a given family of dimensional systems. It can then be said that a physical equation which is valid for any given dimensional system remains valid for all other dimensional systems of the same family.

It still remains however, to relate the natural system of units to some selected set of four reference parameters. Let the four selected parameters

be denoted by symbols $P_1 \bar{U}_1$, $P_2\bar{U}_2$, $P_3\bar{U}_3$, $P_4\bar{U}_4$ where $\bar{U}_1$, $\bar{U}_2$, $\bar{U}_3$, $\bar{U}_4$ are known units in the fixed system. The overbars serve as a reminder that these four reference parameters have been renumbered, and that the new indices 1, 2, 3, 4 have no necessary relation to the first four terms in the original sequence U_1, U_2, --- U_n. In matrix notation we set

$$\begin{Bmatrix} \log P_1 \\ \log P_2 \\ \log P_3 \\ \log P_4 \end{Bmatrix} = \begin{Bmatrix} \log P \end{Bmatrix} \tag{8.11}$$

These four particular parameters can be expressed by the four relevant rows of Eq (8.10) according to the format

$$\begin{Bmatrix} \log P \end{Bmatrix} = [\bar{a}] \begin{Bmatrix} \log M \end{Bmatrix} \tag{8.12}$$

where $[\bar{a}]$ denotes the corresponding 4 x 4 matrix. The only constraint on the manner in which the four parameters may be chosen is that the determinant of this matrix must be non-vanishing, that is, that

$$\left| \bar{a} \right| \neq 0 \tag{8.13}$$

This mild constraint ensures that matrix $[\bar{a}]$ possesses an inverse. Hence we can invert Eq (8.12) to obtain

$$\begin{Bmatrix} \log M \end{Bmatrix} = [\bar{a}]^{-1} \begin{Bmatrix} \log P \end{Bmatrix} \tag{8.14}$$

This result defines the magnitudes of the four fundamental natural units in terms of the four chosen reference parameters, as required.

By substituting Eq (8.14) into (8.10) the latter may readily be recast into the alternative form

$$\begin{Bmatrix} \log N \end{Bmatrix} = [b] \begin{Bmatrix} \log P \end{Bmatrix} \tag{8.15}$$

where the consistency matrix with respect to the new base of the system becomes

$$[b] = [a]\,[\bar{a}]^{-1} \tag{8.16}$$

Eq (8.15) finally defines the magnitudes of all of the natural units included in $\{\log N\}$ in terms of the given magnitudes of the four reference parameters specified in $\{\log P\}$.

Eq. (8.15) can also be expressed in the alternative form

$$\left(\frac{U_i^*}{U_i}\right) = N_i = P_1^{b_{i1}}\; P_2^{b_{i2}}\; P_3^{b_{i3}}\; P_4^{b_{i4}} \tag{8.17}$$

$$i = 1, 2, \text{----}\ n$$

where the coefficients b_{ij} are the elements of the known matrix $[b]$ defined in Eq (8.16). Thus Eq (8.17) defines the size of each consistent natural unit as a function of the magnitudes of the four selected reference parameters.

9. Buckingham's Pi Theorem

Consider an arbitrary physical parameter expressed firstly in fixed units and secondly in natural units. It is the same physical quantity by either mode of description so that we may write

$$P_i\, U_i = P_i^*\, U_i^* \tag{9.1}$$

However, Eq (8.17) shows that

$$U_i^* = N_i\, U_i = P_1^{b_{i1}}\; P_2^{b_{i2}}\; P_3^{b_{i3}}\; P_4^{b_{i4}}\; U_i \tag{9.2}$$

Substituting Eq (9.2) into (9.1), then solving for P_i^* gives

$$P_i^* = \frac{P_i}{N_i} = \frac{P_i}{P_1^{b_{i1}}\, P_2^{b_{i2}}\, P_3^{b_{i3}}\, P_4^{b_{i4}}} \tag{9.3}$$

Notice that the fixed unit U_i cancels out of this result so that P_i^* is a true dimensionless number. The exponents b_{ij} in Eq (9.3) are simply the i th row of the [b] matrix as defined in Eq (8.16). The numerator of Eq (9.3) represents the magnitude of a given physical quantity while the denominator represents the corresponding magnitude of its respective natural unit.

Examination of Eq (9.3) now reveals that it is simply the statement of the well known formula for finding the dimensionaless pi's of Buckingham's celebrated Pi Theorem! While this formula itself is therefore nothing new, what is new is the conceptual framework within which this classical result is now embodied. This provides a new perspective and a deeper insight into the essential meaning of the formula. Specifically, it shows that each dimensionless pi of Buckingham's Pi Theorem represents some corresponding physical parameter as expressed in some appropriate system of consistent natural units! Notice especially that every such quantity retains all of its unique physical character and significance despite the fact that it is now represented in dimensionless form, that is, in a form which is independent of any particular set of fixed units. Moreover, the new framework is far more comprehensive than that usually associated with the Pi Theorem because, instead of being concerned solely with some limited number of fixed parameters which characterize a problem, the system of natural units embraces the phenomena of interest in their totality including both experimental and analytical aspects, and including all constants, parameters, variables and equations which pertain to the situation.

Noting that parameter P_i in Eq (9.3) is arbitrary, let us examine what happens if P_i be chosen to coincide with any one of the four dimensional reference parameters themselves, say for definiteness with P_1 so that

$$P_i = P_1 \tag{9.4}$$

In that case the respective exponents in Eq (9.3) can be shown from the appropriate line of the matrix [b] to reduce simply to

$$b_{i1} = 1 \quad b_{i2} = 0 \quad b_{i3} = 0 \quad b_{i4} = 0 \tag{9.5}$$

Upon substituting Eqs (9.4) and (9.5) into (9.3) that equation simplifies at once to

$$P_1^* = 1 \tag{9.6}$$

The same general procedure can be repeated with the other three reference parameters, with corresponding results. In this way we show finally that when the four reference parameters themselves are expressed in natural units they reduce simply to

$$P_1^* = P_2^* = P_3^* = P_4^* = 1 \tag{9.7}$$

Thus the natural system of units may be defined as one which reduces the numerical measure of all four of the reference parameters to unity. This is accomplished by taking advantage of the four degrees of freedome that are inherent in the natural system of units, namely, the degrees of freedom that correspond to the arbitrary magnitudes of the four fundamental units.

Consider the analytical simplifications that accrue from these circumstances. If we express all equations in natural units, the four reference parameters take on values of unity throughout as indicated by Eq (9.7) and therefore cease to appear explicity in any equation. The net effect is to reduce by four the number of significant parameters involved in the analysis! This, indeed, is what Buckingham's theorem itself tells us. But the present paper helps to make clear the underlying reason for this remarkable result.

Of course, it should be evident that generally parallel results can be developed for systems with either more or fewer than four degrees of freedom and for types of systems other than inertial-mechanical.

10. Concluding Note

The fundamental dimensional principles considered in this paper apply in some form to every quantitative analytical and experimental problem in the entire realm of physical science and engineering. Because of their extraordinary scope and fundamental importance, it is essential to formulate the axiomatic principles of dimensional analysis in terms that are as clear and correct as possible.

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TABLE 2.1 Typical Relations of Consistency In Three Systems of Units

	Physical Dimension	Generalized Units: Basic System (F,M,L,T,H,θ,Q)	Inertial-Mechanical (F,L,T,θ)	Gravitational-Calorimetric (G,L,T,θ)
1.	Angle	1	1	1
2.	Area	L^2	L^2	L^2
3.	Volume	L^3	L^3	L^3
4.	Velocity	L/T	L/T	L/T
5.	Acceleration	L/T^2	L/T^2	L/T^2
6.	Pressure	F/L^2	F/L^2	G/L^2
7.	Moment	FL	FL	GL
8.	Work	FL	FL	GL
9.	Power	FL/T	FL/T	GL/T
10.	Moment of Inertia	ML^2	FLT^2	GL^2
11.	Density	M/L^3	FT^2/L^4	G/L^3
12.	Momentum	ML/T	FT	GL/T
13.	Viscosity	FT/L^2	FT/L^2	GT/L^2
14.	Gas Constant	$FL/M\theta$	$L^2/T^2\theta$	L/θ
15.	Specific Enthalpy	H/M	L^2/T^2	θ
16.	Specific Heat	$H/M\theta$	$L^2/T^2\theta$	1
17.	Specific Entropy	$H/M\theta$	$L^2/T^2\theta$	1
18.	Thermal Conductivity	$H/TL\theta$	$F/T\theta$	G/TL
19.	Electric Current	Q/T	-	-
20.	Electric Potential	FL/Q	-	-

TABLE 8.1 Typical Consistency Matrix for Inertial-Mechanical System

	Physical Dimension	F	L	T	θ
1.	Force	+1	0	0	0
2.	Length	0	+1	0	0
3.	Time	0	0	+1	0
4.	Temperature	0	0	0	+1
5.	Angle	0	0	0	0
6.	Area	0	+2	0	0
7.	Volume	0	+3	0	0
8.	Velocity	0	+1	-1	0
9.	Acceleration	0	+1	-2	0
10.	Pressure	+1	-2	0	0
11.	Moment	+1	+1	0	0
12.	Work	+1	+1	0	0
13.	Power	+1	+1	-1	0
14.	Moment of Inertia	+1	+1	+2	0
15.	Density	+1	-4	+2	0
16.	Momentum	+1	0	+1	0
17.	Viscosity	+1	-2	+1	0
18.	Gas Constant	0	+2	-2	-1
19.	Specific Enthalpy	0	+2	-2	0
20.	Specific Heat	0	+2	-2	-1
21.	Specific Entropy	0	+2	-2	-1
22.	Thermal Conductivity	+1	0	-1	-1

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